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Bosons, Fermions, and the Feshbach-Villars Transformation

JOSÉ VASSALO PEREIRA

Faculdade de Ciências de Lisboa, Portugal[1]

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Abstract

In this paper we develop a general method providing the relativistic equation of wave mechanics for the antiparticles within the frame of the "theory of the fusion" for particles with arbitrary values of spin (de Broglie, 1954). Such a method will enable us to discuss some fundamental differences between bosons and fermions.

1. *Fusion of Corpuscles*

To make acquaintance with the "theory of the fusion" easier for the reader we shall begin by writing the Dirac equation of the particle with spin $\frac{1}{2}$, proper mass m, charge q, and moving in an electromagnetic field given by the potentials $\mathbf{A}$, V, under the form

$$[\epsilon_{op/c} + \mathbf{\Pi}_{op} \cdot \alpha - mc\alpha_4]\Psi = 0 \tag{1.1}$$

where

$$\epsilon_{\text{op}} = (E - qV)_{\text{op}} = -i\hbar\partial_t - qV \qquad \mathbf{\Pi}_{\text{op}} = [\mathbf{p} - ((q/c)\mathbf{A})_{\text{op}} = i\hbar\nabla - (q/c)\mathbf{A} \tag{1.2}$$

Our notation is standard: Ψ is a column matrix with four components and $\boldsymbol{\alpha}(\alpha_1\alpha_2\alpha_3)$, α_4 are four arbitrary Hermitian matrices, with the only restriction being that they obey the well-known conditions

$$\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta\mu\nu \qquad (\mu, \nu = 1,2,3,4) \tag{1.3}$$

In the sequel we shall thus make use of the following choice of the α_μ:

$$\alpha = -\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \alpha_u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.4}$$

[1] On leave at the Fondation Louis de Broglie, Paris.

where σ denotes the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.5}$$

It is well known that the equations (1.1) describe as well the corresponding antiparticle, that is, with the same distinguishing physical constants except for the electric charge, which is now $-q$ instead of q. More precisely, the antiparticle is described by (1.1) with $-q$ in place of q and with a new wave function Ψ_A whose components are unambiguously related to those of Ψ. It will be useful for our purposes to recall briefly the way one can get such relations: Taking (1.2) and (1.4) into account, the Dirac equation (1.1) then becomes

$$(-i\hbar\partial_t - qV - mc^2)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - c\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\sigma}\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = 0$$

$$(-i\hbar\partial_t - qV + mc^2)\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} - c\left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\sigma}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

Let us now take the complex conjugate of the preceding equations. Since, according to (1.5), we have

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^T \tag{1.6}$$

we obtain

$$(i\hbar\partial_t - qV - mc^2)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^* - c\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\sigma}^T\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}^* = 0$$

$$(i\hbar\partial_t - qV + mc^2)\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}^* - c\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\sigma}^T\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^* = 0$$

Again with (1.5) we may easily verify that

$$\sigma_2\boldsymbol{\sigma}^T = -\boldsymbol{\sigma}\sigma_2 \tag{1.7}$$

so that, if we multiply our equations by σ_2, we get

$$(-i\hbar\partial_t + qV - mc^2)\left[-\sigma_2\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}^*\right] - c\left(i\hbar\nabla + \frac{q}{c}\mathbf{A}\right)\cdot\boldsymbol{\sigma}\left[\sigma_2\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^*\right] = 0$$

$$(-i\hbar\partial_t + qV + mc^2)\left[\sigma_2\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^*\right] - c\left(i\hbar\nabla + \frac{q}{c}\mathbf{A}\right)\cdot\left[\sigma_2\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}^*\right] = 0$$

which are the Dirac equations we start from, with $-q$ instead of q and the wave function $\Psi_A = \beta_1\Psi^*$ instead of Ψ, with

$$\beta_1 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{1.8}$$

[According to (1.5), β_1 is not a Hermitian matrix but rather an anti-Hermitian one.]

After this brief digression about a well-known result of the theory of the particle with spin $\frac{1}{2}$, we shall now introduce the equations of the particle of arbitrary spin or, more exactly, of maximum spin $n/2$ (n a positive integer). These will appear to bear an undeniable formal resemblance to those of Dirac, but the wave function has now a greater number of components (4^n for the case of the "fusion" of n corpuscles of spin $\frac{1}{2}$, thus giving rise to a particle with maximum value of spin $n/2$), and the matrices appearing in the equations are now $4^n \times 4^n$ matrices defined by means of Dirac's α_μ. To put it clearly, let us write (1.1) in the form

$$\left[\sum_{\mu=1}^{4} P_\mu \gamma_\mu - imc\right]\Psi = 0 \tag{1.9}$$

where γ_μ ($\mu = 1, 2, 3, 4$) are the four von Neumann matrices

$$\boldsymbol{\gamma} = i\alpha_4 \boldsymbol{\alpha}, \qquad \gamma_4 = \alpha_4 \tag{1.10}$$

and

$$P_\mu = p_\mu - (q/c)\phi_\mu \tag{1.11}$$

is a combination of the two 4-vectors

$$p_\mu = i\hbar\partial_\mu \qquad (x_4 = ict) \tag{1.12}$$

$$\phi_\mu = [\mathbf{A}, iV] \tag{1.13}$$

Now the equations of the particle of proper mass m and electric charge q, arising from the "fusion" of n Dirac corpuscles that is, a particle that is able to assume any one of the values $n/2, n/2\text{-}1, n/2\text{-}2, \ldots, [1-(-1)^n]/4$ of the spin have a form similar to that of (1.9):

$$\left[\sum_{\mu=1}^{4} P_\mu \mathsf{P}_\mu - imc \;\; \Psi = 0\right] \tag{1.14}$$

In this equation, P_μ has the meaning of (1.11). As for the Ψ, it is a column matrix with 4^n elements (instead of 4, as was the case in Dirac's theory), and the P_μ are four $4^n \times 4^n$ matrices defined by means of the γ_μ's (and thus of the α_μ's) in the following way:

$$\mathsf{P}_\mu = (1/n)[\overbrace{(\underbrace{\gamma_\mu \times 1 \times \cdots \times 1}_{n}) + (\underbrace{1 \times \gamma_\mu \times \cdots \times 1}_{n}) + \cdots + (\underbrace{1 \times 1 \times \cdots \times \gamma_\mu}_{n})}^{n}],$$

$$(\mu = 1, 2, 3, 4) \tag{1.15}$$

In this formula 1 is the identity matrix 4×4 and $A \times B$ denotes the exterior product of matrices $A = (a_{it})$ and $B = (b_{it})$ which is a matrix obtained by replacing in A the element a_{ik} by the matrix $a_{ik}B$. Evidently, $A \times B \neq B \times A$

and $(A \times B) \times C = A \times (B \times C) = A \times B \times C$. We recall some more properties of this definition that will be needed later:

(a) $A \times (kB) = (kA) \times B = k(A \times B)$ (k const)

(b) $(A_1 \times A_2 \times \cdots \times A_p)(B_1 \times B_2 \times \cdots \times B_p) = (A_1B_1) \times (A_2B_2) \times (\cdots) \times (A_pB_p)$ (1.16)

(c) $(A_1 \times A_2 \times \cdots \times A_p)^T = A_1{}^T \times A_2{}^T \times \cdots A_p{}^T$ (the symbol T denotes the transpose of a matrix)

Thus (1.14) and (1.15) provide the relativistic equations of the particle of charge q, proper mass m, and spin maximum $n/2$, arising from the "fusion" of n corpuscles of Dirac. These equations have been given (de Broglie, 1954; Petiau, 1953) for the case of non charged particles. The generalization we give here seems in all ways natural and will enable us to describe the particle as well as the antiparticle. The original aim of de Broglie with the theory of fusion was the theory of light (de Broglie, 1934, 1952), but soon the method was developed for arbitrary values of spin (de Broglie, 1954; Petiau, 1947, 1952).

2. *Antiparticles and Spin Maximum 1*

To start with, let us study the case of the fusion of two corpuscles of Dirac. The equations of the particle of spin maximum 1 and charge q are given by (1.14) and (1.15) with $n = 2$. The wave function Ψ is now a column matrix with $4^2 = 16$ components that may be labeled as follows

$$\Psi^T = (\psi_{11}\psi_{12}\psi_{13}\psi_{14}\psi_{21}\psi_{22} \cdots \psi_{43}\psi_{44}) \tag{2.1}$$

According to (1.10) and (1.16b) matrices P_μ now take the form

$$\mathsf{P} = (i/2)[(\alpha_4 \times 1)(\boldsymbol{\alpha} \times 1) + (1 \times \alpha_4)(1 \times \boldsymbol{\alpha})], \qquad \mathsf{P}_4 = \tfrac{1}{2}[(\alpha_4 \times 1) + (1 \times \alpha_4)]$$

At this stage we are going to introduce a usual and very useful notation in the theory of fusion, by defining the following matrices:

$$a_\mu = \alpha_\mu \times 1, \qquad b_\mu = 1 \times \alpha_\mu \qquad (\mu = 1, 2, 3, 4) \tag{2.2}$$

With this notation we have

$$\mathsf{P} = (i/2)(a_4\mathbf{a} + b_4\mathbf{b}), \qquad \mathsf{P}_4 = \tfrac{1}{2}(a_4 + b_4)$$

so that the equations of the charged particle with spin maximum 1 can be written

$$\left[\frac{1}{2c}(-i\hbar\partial_t - qV)(a_4 + b_4) + \frac{1}{2}\left(i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right)\cdot(a_4\mathbf{a} + b_4\mathbf{b}) - mc\right]\Psi = 0 \tag{2.3}$$

The properties of the matrices a_μ and b_μ are similar to those of the α_μ. In fact, one may easily verify by means of (2.2), (1.15b), and (1.3) that

$$a_\mu b_\nu = b_\nu a_\mu, \qquad a_\mu a_\nu + a_\nu a_\mu = b_\mu b_\nu + b_\nu b_\mu = 2\delta_{\mu\nu} \qquad (\mu, \nu = 1, 2, 3, 4) \tag{2.4}$$

According to this, if we now multiply (2.3) on the left by $a_4 b_4$, we get

$$\left[\frac{1}{2c}(-i\hbar\partial_t - qV)(a_4 + b_4) + \frac{1}{2}\left(i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right)\cdot(\mathbf{a}b_4 + a_4\mathbf{b}) - mca_4b_4\right]\Psi = 0 \tag{2.5}$$

Though (2.3) and (2.5) are obviously mathematically equivalent, the preceding form is the more suitable for our aims since all the matrices appearing in it are Hermitian, which is not the case of $a_4\mathbf{a} + b_4\mathbf{b}$ in (2.3).

Starting from (2.5) let us now search the equations of the corresponding antiparticle. Taking the complex conjugate we get

$$\left[\frac{1}{2c}(i\hbar\partial_t - qV)(a_4 + b_4)^* + \frac{1}{2}\left(-i\hbar\,\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}\right)\cdot(\mathbf{a}b_4 + a_4\mathbf{b})^* - mca_4^*b_4^*\right]\Psi^* = 0$$

Let us now assume that a matrix 16 x 16, β_2, exists such that

$$\begin{aligned}\beta_2(a_4 + b_4)^* &= -(a_4 + b_4)\beta_2\\ \beta_2(\mathbf{a}b_4 + a_4\mathbf{b})^* &= -(\mathbf{a}b_4 + a_4\mathbf{b})\beta_2\\ \beta_2 a_4^* b_4^* &= a_4 b_4 \beta_2\end{aligned} \tag{2.6}$$

If one multiplies our equations by β_2 they become

$$\left[\frac{1}{2c}(-i\hbar\partial_t + qV)(a_4 + b_4) + \frac{1}{2}\left(i\hbar\boldsymbol{\nabla} + \frac{q}{c}\mathbf{A}\right)\cdot(\mathbf{a}b_4 + a_4\mathbf{b}) - mca_4b_4\right]\beta_2\Psi^* = 0$$

which are precisely the equations we start from, with q replaced by $-q$ and Ψ by $\beta_2\Psi^*$. So all we must do is to find a matrix β_2 fulfilling the conditions required above. We are going to show that

$$\beta_2 = \beta_1 \times \beta_1$$

where β_1 is the 4 x 4 matrix we met already in the Dirac theory and defined in (1.8). We may start by demonstrating some properties of the matrix β_1 that will be needed later. We have, by (1.4) and (1.6)-(1.8),

$$\beta_1\boldsymbol{\alpha}^* = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}^T \\ \boldsymbol{\sigma}^T & 0 \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}\sigma_2 & 0 \\ 0 & \boldsymbol{\sigma}\sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$\beta_1\alpha_4^* = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

In short,

$$\beta_1\boldsymbol{\alpha}^* = \boldsymbol{\alpha}^*\beta_1 \qquad\qquad \beta_1\alpha_4^* = -\alpha_4\beta_1 \tag{2.7}$$

The preceding formulas enable us to prove (2.6). In fact, by means of (1.16b) and (2.2), we get

$$\begin{aligned}\beta_2(a_4^* + b_4^*) &= (\beta_1\alpha_4^* \times \beta_1) + (\beta_1 \times \beta_1\alpha_4^*) = -(\alpha_4\beta_1 \times \beta_1) - (\beta_1 \times \alpha_4\beta_1)\\ &= -[(\alpha_4 \times 1) + (1 \times \alpha_4)](\beta_1 \times \beta_1) = -(a_\mu + b_\mu)\beta_2\end{aligned}$$

and, similarly,

$$\beta_2(\mathbf{a}b_4 + a_4\mathbf{b})^* = (\beta_1\boldsymbol{\alpha}^* \times \beta_1\alpha_4^*) + (\beta_1\alpha_4^* \times \beta_1\boldsymbol{\alpha}^*) = -(\boldsymbol{\alpha}\beta_1 \times \alpha_4\beta_1) - (\alpha_4\beta_1 \times \beta_1)$$
$$= -[(\boldsymbol{\alpha} \times 1)(1 \times \alpha_4) - (\alpha_4 \times 1)(1 \times \boldsymbol{\alpha})](\beta_1 \times \beta_1) = -(\mathbf{a}b_4 + a_4\mathbf{b})\beta_2$$

$$\beta_2 a_4^* b_4^* = \beta_1\alpha_4^* \times \beta_1\alpha_4^* = \alpha_4\beta_1 \times \alpha_4\beta_1 = (\alpha_4 \times 1)(1 \times \alpha_4)(\beta_1 \times \beta_1)$$
$$= a_4 b_4 \beta_2$$

in total agreement with (2.6). It then follows from the preceding demonstration that if (2.5) describes the particle of spin maximum 1, mass m, and charge q, the same equation with q replaced by $-q$ and Ψ by

$$\Psi_A = \beta_2\Psi^* = (\beta_1 \times \beta_1)\Psi^* \tag{2.8}$$

describes the antiparticle. According to (1.5), (1.8), and (2.1) is follows that the whole theory of the antiparticle of spin maximum 1 can be obtained from the theory of the corresponding particle by simply replacing q by $-q$ and the components ψ_{ik} of Ψ in (2.1) by

$$\Psi_A{}^T = (-\psi_{44}^*, \psi_{43}^*, \psi_{42}^*, -\psi_{41}^*, \psi_{34}^*, -\psi_{33}^*, -\psi_{32}^*, \psi_{31}^*, \psi_{24}^*, -\psi_{23}^*, -\psi_{22}^*, \psi_{21}^*,$$
$$-\psi_{14}^*, \psi_{13}^*, \psi_{12}^*, -\psi_{11}^*) \tag{2.9}$$

3. *The General Case*

The same result can still be domonstrated in the general case of the fusion of n corpuscles of spin $\frac{1}{2}$, as we are now going to show. The equations of the particle of spin maximum $n/2$ are given by (1.14). According to (1.10), (1.15), and (1.16b), matrices P_μ now take the form

$$\mathbf{P} = \frac{i}{n}[(\alpha_4 \times 1 \times \cdots \times 1)(\boldsymbol{\alpha} \times 1 \times \cdots \times 1) + (1 \times \alpha_4 \times \cdots \times 1)(1 \times \boldsymbol{\alpha} \times \cdots \times 1)$$
$$+ \cdots + (1 \times \cdots \times \alpha_4)(1 \times \cdots \times \boldsymbol{\alpha})]$$
$$\mathsf{P}_n = (1/n)[(\alpha_4 \times 1 \times \cdots \times 1) + (1 \times \alpha_4 \times \cdots \times 1) + \cdots + (1 \times \cdots \times \alpha_4)]$$

Formulas (2.2) now obviously suggest the following definitions:

$$\begin{aligned} a_\mu &= \alpha_\mu \times 1 \times 1 \times \cdots \times 1 \\ b_\mu &= 1 \times \alpha_\mu \times 1 \times \cdots \times 1 \\ c_\mu &= 1 \times 1 \times \alpha_\mu \times \cdots \times 1 \qquad \mu = 1, 2, 3, 4 \\ &\vdots \\ h_\mu &= 1 \times 1 \times 1 \times \cdots \times \alpha_\mu \end{aligned} \tag{3.1}$$

Here again we may ascertain without difficulties that the properties of the $a_\mu, b_\mu, \cdots, h_\mu$ are very similar to those of the Dirac's α_μ. More precisely,

$$\begin{aligned} a_\mu a_\nu + a_\nu a_\mu = b_\mu b_\nu + b_\nu b_\mu = \cdots = h_\mu h_\nu + h_\nu h_\mu = 2\delta_{\mu\nu} \\ a_\mu b_\nu = b_\nu a_\mu, \qquad a_\mu c_\nu = c_\nu a_\mu, \qquad b_\mu c_\nu = c_\nu b_\mu, \text{ etc.} \end{aligned} \tag{3.2}$$

The matrices appearing in the equations are then

$$\mathsf{P} = (1/n)(a_4\mathbf{a} + b_4\mathbf{b} + \cdots + h_4\mathbf{h}) \qquad \mathsf{P}_4 = (1/n)(a_4 + b_4 + \cdots + h_4)$$

and the equations themselves take the form

$$\left[\frac{1}{nc}(-i\hbar\partial_t - qV)(a_4 + b_4 + \cdots + h_4) + \left(i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)\cdot\frac{1}{n}(a_4\mathbf{a} + b_4\mathbf{b} + \cdots + h_4\mathbf{h}) - mc\right]\Psi = 0 \tag{3.3}$$

where Ψ is a column matrix with 4^n components. Multiplying on the left by $a_4b_4c_4\cdots h_4$, we get, according to (3.2),

$$\left[\frac{1}{nc}(-i\hbar\partial_t - qV)(b_4c_4\cdots h_4 + a_4c_4\cdots h_4 + \cdots + a_4b_4\cdots g_4)\right.$$
$$\left. + \left(i\hbar \quad - \frac{q}{c}\mathbf{A}\right)\cdot\frac{1}{n}(\mathbf{a}b_4c_4\cdots h_4 + a_4\mathbf{b}c_4\cdots h_4 + \cdots + a_4b_4\cdots\mathbf{h}) - mca_4b_4\cdots h_4\right]\Psi = 0 \tag{3.4}$$

In order to obtain from these equations those of the associated antiparticle, we take their complex conjugate and multiply on the left by a $4^n \times 4^n$ matrix, β_n, such that

$$\begin{aligned} \beta_n(b_4c_4\cdots h_4 + a_4c_4\cdots h_4 + \cdots + a_4b_4\cdots g_4)^* &= k_n(b_4c_4\cdots h_4 + \cdots + a_4b_4\cdots g_4)\beta_n \\ \beta_n(\mathbf{a}b_4c_4\cdots h_4 + a_4\mathbf{b}c_4\cdots h_4 + \cdots + a_4b_4\cdots\mathbf{h})^* &= k_n(\mathbf{a}b_4\cdots h_4 + \cdots + a_4b_4\cdots\mathbf{h})\beta_n \\ \beta_n(a_4b_4\cdots h_4)^* &= -k_n(a_4b_4\cdots h_4)\beta_n \end{aligned} \tag{3.5}$$

where k_n is a constant to be determined. If such a β_n exists then the equations (3.4) go into the form

$$\left[\frac{1}{nc}(-i\hbar\partial_t + qV)(b_4c_4\cdots h_4 + \cdots + a_4b_4\cdots g_4)\right.$$
$$\left. + \left(i\hbar\nabla + \frac{q}{c}\mathbf{A}\right)\cdot\frac{1}{n}(\mathbf{a}b_4c_4\cdots h_4 + \cdots + a_4b_4\cdots\mathbf{h}) - mca_4b_4\cdots h_4\right]\beta_n\Psi^* = 0$$

which is the original form of the equations with q replaced by $-q$ and Ψ by

$$\Psi_A = \beta_n\Psi^* \tag{3.6}$$

The relativistic equations of the particle of spin maximum $n/2$ and charge q thus allow us to obtain those of the antiparticle. It still remains to find β_n fulfilling the conditions (3.5). It can easily be seen that

$$\beta_n = \underbrace{\beta_1 \times \beta_1 \times \cdots \times \beta_1}_{n} \tag{3.7}$$

In fact,

$$\beta_n(b_4c_4 \cdots h_4 + a_4c_4 \cdots h_4 + \cdots + a_4b_4 \cdots g_4)^* = (\beta_1 \times \beta_1\alpha_4^* \times \beta_1\alpha_4^* \times \cdots \times \beta_1\alpha_4^*)$$
$$+ (\beta_1\alpha_4^* \times \beta_1 \times \beta_1\alpha_4^* \times \cdots \times \beta_1\alpha_4^*) + \cdots + (\beta_1\alpha_4^* \times \beta_1\alpha_4^* \times \cdots \times \beta_1)$$

According to (2.7) this amounts to saying that

$$(-1)^{n-1}[(1 \times \alpha_4 \times \cdots \times 1)(1 \times 1 \times \alpha_4 \times \cdots \times 1) \cdots (1 \times 1 \times \cdots \times \alpha_4)$$
$$+ (\alpha_4 \times 1 \times \cdots \times 1)(1 \times 1 \times \alpha_4 \times \cdots \times 1) \cdots (1 \times 1 \times \cdots \times \alpha_4) + \cdots$$
$$+ (\alpha_4 \times 1 \times \cdots \times 1)(1 \times \alpha_4 \times \cdots \times 1) \cdots (1 \times 1 \times \cdots \times \alpha_4 \times 1)]\,(\beta_1 \times \beta_1 \times \cdots \times \beta_1)$$
$$= (-1)^{n-1}(b_4c_4 \cdots h_4 + a_4c_4 \cdots h_4 + a_4b_4 \cdots g_4)\beta_n$$

By a similar calculation one could ascertain that

$$\beta_n(\mathbf{a}b_4c_4 \cdots h_4 + a_4\mathbf{b}c_4 \cdots h_4 + \cdots + a_4b_4 \cdots \mathbf{h})^*$$
$$= (-1)^{n-1}(\mathbf{a}b_4 \cdots h_4 + \cdots + a_4b_4 \cdots \mathbf{h})\beta_n$$

Finally, we have

$$\beta_n(a_4b_4 \cdots h_4)^* = \beta_1\alpha_4^* \times \beta_1\alpha_4^* \times \cdots \times \beta_1\alpha_4^* = (-1)^n(\alpha_4\beta_1 \times \alpha_4\beta_1 \times \cdots \times \alpha_4\beta_1)$$
$$= (-1)^n(\alpha_4 \times 1 \times \cdots \times 1)(1 \times \alpha_4 \times \cdots \times 1) \cdots (1 \times \cdots \times \alpha_4)$$
$$\times (\beta_1 \times \beta_1 \times \cdots \times \beta_1)$$
$$= (-1)^n a_4b_4c_4 \cdots h_4\beta_n$$

We thus have proved that the conditions (3.5) are fulfilled by β_n given by (3.7) and

$$k_n = (-1)^{n-1} \tag{3.8}$$

4. *Probability Density and Probability Current Vector of the Antiparticles*

As is well known, (1.1) allows us to establish an equation of the form

$$\partial_t\rho + \nabla \cdot \mathbf{j} = 0 \tag{4.1}$$

with

$$\rho = \Psi^+\Psi \qquad \mathbf{j} = -c\Psi^+\boldsymbol{\alpha}\Psi$$

(A^+ obviously means A^{*T}). The theory of Dirac assigns to ρ and $\mathbf{j}$ the physical meaning of a probability density and a probability current vector. ρ and $\mathbf{j}$ are real and since ρ is positive definite, one may expect that the same will happen with the probability density of the antiparticle. In fact, we have

$$\rho_A = \Psi_A^+\Psi_A = (\beta_1\Psi^*)^+(\beta_1\Psi^*) = \Psi^T\beta_1^+\beta_1\Psi^*$$

As has been pointed out above, β_1 is an anti-Hermitian matrix, that is,

$$\beta_1^+ = -\beta_1 \tag{4.2}$$

besides, we have, according to (1.8),

$$\beta_1{}^2 = -1 \tag{4.3}$$

The expression of ρ_A then becomes

$$\rho_A = -\Psi^T\beta_1{}^2\Psi^* = \Psi^+\Psi = \rho$$

If we consider the probability current vector $\mathbf{j}$ we get for the antiparticle

$$\mathbf{j}_A = -c(\beta_1\Psi^*)^+\boldsymbol{\alpha}(\beta_1\Psi^*) = -c\Psi^T\beta_1^+\boldsymbol{\alpha}\beta_1\Psi^*$$

By means of (2.7), (4.2), and (4.3) we obtain

$$\mathbf{j}_A = -c\Psi^T\boldsymbol{\alpha}^*\Psi^* = -c(\Psi^+\boldsymbol{\alpha}\Psi)^* = \mathbf{j}$$

Thus in the theory of Dirac the probability density and current vector have the same values for the particle and for the antiparticle. The probability density is then positive definite in both cases. It will become clear in the sequel that that fact is inherent in the Hamiltonian structure of the field equations; therefore it may not happen when dealing with particles obtained by fusion of several corpuscles of Dirac, whose equations, as we have seen, are not of the form $i\hbar\partial_t\Psi = H\Psi$. Besides, if, on one hand, they still allow us to define ρ and $\mathbf{j}$ obeying the continuity equation (4.1), the other hand is no longer positive definite. [About the physical meaning to be given to a "probability" density that is not positive definite see de Broglie (1954, 1963). We do not discuss this problem here.] In fact, and taking the simple case of $n = 2$ (fusion of two corpuscles), one could easily verify that the equations (2.5) of the particle with maximum spin 1 imply an equation of continuity (4.1) with

$$\rho = \Psi^+\frac{a_4 + b_4}{2}\Psi, \qquad \mathbf{j} = -c\Psi^+\frac{\mathbf{a}b_4 + a_4\mathbf{b}}{2}\Psi \tag{4.4}$$

Since the matrices entering in these expressions are Hermitian, ρ and $\mathbf{j}$ are real. Nevertheless, we see that ρ is not positive definite. What are now the relations between the values of ρ and $\mathbf{j}$ for the particle and the antiparticle? We have, by means of (2.6) and (2.8),

$$\rho_A = (\beta_2\Psi^*)^+\frac{a_4 + b_4}{2}(\beta_2\Psi^*) = \Psi^T\beta_2^+\frac{a_4 + b_4}{2}\beta_2\Psi^* = -\Psi^T\beta_2^+\beta_2\left(\frac{a_4 + b_4}{2}\right)^*\Psi^*$$

Now (2.8), (4.2), and (4.3) imply

$$\beta_2^+ = \beta_2, \qquad \beta_2^{\,2} = 1$$

and ρ_A then becomes

$$\rho_A = -\Psi^T \left(\frac{a_4 + b_4}{2}\right)^* \Psi^* = -\rho$$

The same happens with j, by making use of (2.6):

$$\mathbf{j}_A = -c(\beta_2\Psi^*) + \frac{\mathbf{a}b_4 + a_4\mathbf{b}}{2}(\beta_2\Psi^*) = c\Psi^T\beta_2^+\beta_2\left(\frac{\mathbf{a}b_4 + a_4\mathbf{b}}{2}\right)^* \Psi^* = -\mathbf{j}$$

Such a result is not surprising. One has but to remember that when we take the equation of Klein-Gordon with a single component wave function Ψ to describe the particle with spin 0 (which is, in the theory of the fusion, one of the two possible realizations of the particle with maximum spin 1), the same equation with q and Ψ replaced by $-q$ and Ψ^* then describes the antiparticle. Hence $\rho_A = -\rho$ and $\mathbf{j}_A = -\mathbf{j}$ (they are obviously not positive definite).

From the above results we may predict what will happen in the fusion of an arbitrary number of Dirac corpuscles. Let us take, for instance, the case of the fusion of three corpuscles, giving rise to a particle whose spin may have one of the two values 3/2 or 1/2; since the resulting particle may thus be the Dirac corpuscle itself for which, as we have seen, the relations $\rho_A = \rho$, $\mathbf{j}_A = \mathbf{j}$ are valid we may then expect the same property to be valid for a particle of maximum spin 3/2. The same can be said for the case of a fusion of an odd number of Dirac corpuscles since in this case there is always, among the possible values of the spin of the resulting particle, the value $\frac{1}{2}$. This must then be an essential feature of the particles with half-integer spin. The preceding considerations can be easily transposed to the case of a fusion of an even number of Dirac corpuscles. In this case, the unity is always one of the possible values of the spin of the particle arising from the fusion and since for that value we have $\rho_A = -\rho$, $\mathbf{j}_A = -\mathbf{j}$ it is likely enough that the same must be true for the general case of n odd. In order to verify what we have said, we may begin by remarking that we have

$$\beta_n^+ = (-1)^n\beta_n, \qquad \beta_n^{\,2} = (-1)^n \tag{4.5}$$

which is an immediate consequence of (3.7), (4.2), and (4.3). Now it follows from the general equations (3.4) of the particle with maximum spin $n/2$ that an equation of continuity (4.1) is valid with the following real expressions for ρ and j:

$$\begin{aligned} \rho &= \frac{1}{n}\Psi^+(b_4c_4\cdots h_4 + a_4c_4\cdots h_4 + \cdots + a_4b_4\cdots g_4)\Psi \\ \mathbf{j} &= -\frac{c}{n}\Psi^+(\mathbf{a}b_4\cdots h_4 + a_4\mathbf{b}\cdots h_4 + \cdots + a_4b_4\cdots \mathbf{h})\Psi \end{aligned} \tag{4.6}$$

For the corresponding antiparticle we get, by means of (3.5), (3.6), and (4.5),

$$\rho_A = \frac{1}{n}\Psi_A^+(b_4c_4\cdots h_4 + \cdots + a_4b_4\cdots g_4)\Psi = \frac{1}{n}\Psi^T\beta_n^+(\cdots)\beta_n\Psi^*$$
$$= \frac{1}{n}\Psi^T\frac{\beta_n^+\beta_n}{k_n}(\cdots)^*\Psi^* = \frac{1}{nk_n}\Psi^T(\cdots)^*\Psi^*$$

that is, according to (3.8),

$$\rho_A = \frac{(-1)^{n-1}}{n}(\Psi^+(\cdots)\Psi)^* = (-1)^{n-1}\rho$$

In a similar way, we find for $\mathbf{j}_A$

$$\mathbf{j}_A = -\frac{c}{n}\Psi_A^+(\mathbf{a}b_4\cdots h_4 + \cdots + a_4b_4\cdots \mathbf{h})\Psi_A = -\frac{c}{n}\Psi^T\beta_n^+(\overrightarrow{\cdots})\beta_n\Psi^*$$
$$= -\frac{c}{n}\Psi^T\frac{\beta_n^+\beta_n}{k_n}(\overrightarrow{\cdots})^*\Psi^* = -\frac{c}{n}(-1)^{n-1}(\Psi^+(\overrightarrow{\cdots})\Psi)^* = (-1)^{n-1}\mathbf{j}$$

In brief, we have

$$\rho_A = (-1)^{n-1}\rho, \qquad \mathbf{j}_A = (-1)^{n-1}\mathbf{j} \tag{4.7}$$

The theory of the fusion thus assigns to bosons and antibosons symmetric values for the expressions of the probability density and the probability current vector, while for fermions and antifermions such values are the same.

5. *Spin Zero*

To close the considerations of the preceding paragraphs we want to point out some aspects of the theory of the spin-0 particle as it appears given by the method of the fusion (that is, for instance, as one of the possible cases of the fusion of two Dirac corpuscles) and as it is also often envisaged, that is, as a consequence of a field equation with Hamiltonian form (Feshbach and Villars, 1958).

Before entering into the subject let us recall that in the general theory of the fusion of n corpuscles of spin $\frac{1}{2}$ with the field equations (3.4) it is always possible to define unambiguously certain simple linear combinations of the 4^n spinorial components ψ_{ik} of the wave function Ψ (we denote such expressions by φ_a), such that the field equations written with the φ_a become separated, that is, more precisely, the system of equations (3.4) then divide up in a certain number of subsystems such that the field variables φ_a appearing in each one of these subsystems do not appear in any of the others. Each of these independent subsystems (with the field variables of its own) then describes a unique particle corresponding to one and only one of the possible values of the spin: $n/2, n/2 - 1, \ldots, [1 - (-1)^n]/4$. We may perhaps add that if n is even (odd) the φ_a are tensorial (spinorial) variables. It can then be shown that any physical observable [in particular, ρ and $\mathbf{j}$ given by (4.6)], when written in the new variables φ_a, become separated as well, that is, the expression of any observable is then given by the sum of several terms, each one being a function of those φ_a associated to one of the possible values of the spin arising in the fusion. For example, and taking the simple case of ρ and $n = 2$, we have

$$\rho(\psi_{ik}) = \rho(\psi_{ik}(\varphi_a)) = \rho_1(\varphi_a{}^1) + \rho_0(\varphi_a{}^0)$$

where $\rho(\psi_{ik})$ has the expression (4.6) and $\varphi_a{}^1$ and $\varphi_a{}^0$ denote the tensorial field variables describing the particle of spin 1 and that of spin 0. It seems then justified to say, as is usual in the fusion method, that the theory of a particle with only one of the possible values of the spin allowed by the fusion of n Dirac corpuscles is then obtained from the general theory of the particle with maximum spin $n/2$ by there setting equal to zero all the field variables φ_a corresponding to the other values of the spin and thus keeping only the φ_a concerning the value of the spin we intend. The preceding and somewhat compact considerations have been presented and treated in detail (de Broglie, 1952, 1954) and we do not develop them here.

Now it can be seen (we omit this easy but tedious calculation) that the theory of the particle with spin maximum 1 assigns to the particle of spin 0 five field variables φ_a [each one being a well-defined simple linear function of the ψ_{ik} of (2.1)] : an invariant I and a 4-vector $[\mathbf{S}, iS_4]$ obeying the following equations ($x_4 = ict$):

$$(\epsilon_{\text{op}}/c)I + mcS_4 = 0, \qquad \mathbf{\Pi}_{\text{op}}I + mc\mathbf{S} = 0$$
$$\mathbf{\Pi}_{\text{op}} \cdot \mathbf{S} - (\epsilon_{\text{op}}/c)S_4 = mcI \tag{5.1}$$

[the operators are those defined in (1.2)] . Equations (5.1) are linear, symmetric (that is, with derivatives of first order in space and time), and non-Hamiltonian, and the existence of five field variables points to some interesting consequences: If we had no physical reasons to make use of the 3-vector $\mathbf{S}$, that is, if the theory had only two field variables (I and S_4), and equations (5.1) had the form

$$(\epsilon_{\text{op}}/c)I = -mc^2S_4$$
$$\epsilon_{\text{op}}S_4 + (\mathbf{\Pi}_{\text{op}}^2/m)I + mc^2I = 0 \tag{5.2}$$

(with derivatives of second order in space and first order in time), they would then have the Hamiltonian form. In order to put them into a more symmetrical form we may define

$$\chi_1 = \frac{I - S_4}{\sqrt{2}}, \qquad \chi_2 = \frac{I + S_4}{\sqrt{2}}$$

so that (5.2) becomes

$$\left[\frac{\mathbf{\Pi}_{\text{op}}^2}{2m}(\sigma_3 + i\sigma_2) + mc^2\sigma_3 - \epsilon_{\text{op}}\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = 0$$

where σ_2, σ_3 are Pauli matrices (1.5). This is precisely what is usually done (Feshbach and Villars, 1958). But in this case it is easily seen that $\rho = -\rho_A$ $\mathbf{j} = +\mathbf{j}_A$, in contrast to what happens when, as is the case in the method of fusion, the theory imposes the non-Hamiltonian form of the field equations and the introduction of an additional 3-vector **S** tied to S_4 by means of the 4-vector $[\mathbf{S}, iS_4]$.

Moreover, according to equations (5.1), the method of fusion provides a justification for the purely formal procedure of making use of other field variables in order to give to the Klein-Gordon equation,

$$[\epsilon_{\text{op}}^2/c^2 - \mathbf{\Pi}_{\text{op}}^2 - m^2c^2]\Psi = 0 \tag{5.3}$$

(Ψ is a wave function with a single component), the form

$$\left[\mathbf{\Pi}_{\text{op}} \cdot \boldsymbol{\beta} + \frac{i}{c}\beta_4\epsilon_{\text{op}} + imc\right]\Phi = 0 \tag{5.4}$$

where Φ is a column matrix with five components. In the preceding equation $\boldsymbol{\beta}$, β_4 are the four well-known Kemmer-Petiau matrices $[(\beta_1)_{15} = (\beta_1)_{51} = (\beta_2)_{25} = (\beta_2)_{52} = (\beta_3)_{35} = (\beta_3)_{53} = (\beta_4)_{54} = (\beta_4)_{45}$, other elements null]. The transition from (5.3) to (5.4) is accomplished by introducing the 4-vector $[\psi, \psi_4]$,

$$\boldsymbol{\Psi} = (i/mc)\mathbf{\Pi}_{\text{op}}\Psi \qquad \psi_4 = -(1/mc^2)\epsilon_{\text{op}}\Psi$$

and we have

$$\Phi^T = (\psi_1\psi_2\psi_3\psi_4\Psi)$$

Now it is evident that equations (5.1) are precisely those of Kemmer-Petiau with the notations $\Psi = I$, $\boldsymbol{\Psi} = -i\mathbf{S}$, $\psi_4 = S_4$. It is still evident that equations (5.4) with q replaced by $-q$ and $(\psi_1\psi_2\psi_3\psi_4\Psi)$ by $(\psi_1^*\psi_2^*\psi_3^* - \psi_4^*\Psi)$ describe the antiparticle, and since they imply an equation of continutity (4.1) with

$$\rho = -\frac{h}{2mc}\Psi^*\psi_4 + \text{c.c.} \qquad \mathbf{j} = -\frac{ih}{2m}\Psi^*\boldsymbol{\psi} + \text{c.c.}$$

(c.c. means complex conjugate), we have $\rho = -\rho_A$, $\mathbf{j} = -\mathbf{j}_A$.

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